Direct calculation of invariant measures for chaotic maps

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We show how to construct transfer matrices equivalent to arbitrary nonlinear maps. The eigenvector corresponding to the eigenvalue one gives directly the invariant measure of the map. When applied to the logistic map, this method yields features very similar to those found by histogram methods. We discuss ways to accelerate convergence of results. [S1063-651X(96)51305-X]

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Chaotic systems are defined by sensitive dependence to initial conditions, which makes trajectories unpredictable after long times if the initial condition is not perfectly known. On the other hand, chaotic iterations have robust statistical properties such as the invariant measure [IM or $\rho(x)$], a probability density over the range R of the chaotic variable which does not change if a chaotic map T acts on it. Lorenz says that individual trajectories are analogous to weather, while long-term global properties such as the IM are analogous to climate [1]. The invariant measure satisfies the functional equation $T[\rho(x)]=\rho(x)$. Important properties of a map such as the Lyapunov exponent (λ) can be calculated from the invariant measure: $\lambda=\int_R \ln|T'(x)|\rho(x)dx$.

Analytical expressions for the IM are rare: some have been found [2] for special cases of the logistic, cusp and tent maps starting from the Frobenius-Perron equation, or for piecewise linear maps using a transfer matrix (TM) method. More often the map is assumed to be ergodic (in general, difficult to prove) and a single trajectory of the map is used to collect histogram statistics for the IM: see [3] for example. In this paper we show how to construct transfer matrices for arbitrary chaotic maps. We also show that a particular eigenvector of this matrix gives the IM directly. We then apply our construction to the logistic map, $x_{n+1}=ax_n(1-x_n)$, known to have nontrivial IMs for many values of a, and compare results with the standard histogram method.

As in the histogram approach, we treat $\rho(x)$ as a function of a discrete variable: we divide the relevant range of x into m consecutive intervals, not necessarily of equal widths. Accordingly, the IM will consist of m numbers with the usual properties of a probability distribution. We will subsequently express these numbers as a column vector $\mathbf{r}$. Once we have chosen an appropriate partition, we transform the nonlinear map $T(x)$ into an $m\times m$ matrix $\mathbf{T}$ in the following way.

Figure 1 shows a portion of a typical nonlinear map; we consider the iterates of a partition $h_i=[x_0,x_3]$ of width $h=x_3-x_0$. We assume that the IM is constant *within* each partition [4], and that the TM elements which give the probability of a transition to the partitions $h_j,h_{j+1},\ldots$ are therefore proportional to the width of their preimages in the partition h_i. For the example in Fig. 1, $T_{j,i}=(x_1-x_0)/h$, $T_{j+1,i}=(x_2-x_1)/h$ and $T_{j+2,i}=(x_3-x_2)/h$. We note that the elements $T_{j,i-1},T_{j+2,i+1}\neq 0$, and that for this construction we have $\Sigma_i T_{j,i}=1$ for each column j; this is the definition of a stochastic matrix.

The transfer matrix $\mathbf{T}$ is similar in structure to the map T. For instance, for a parabolic map the nonzero elements of $\mathbf{T}$ will form a parabola of finite width. We note that this method has been used to find $\rho(x)$ analytically for piecewise linear maps [2,5]. The method, however, was not used for arbitrary nonlinear maps. A similar approach was also developed by Fox [6], who formulated a master equation for the

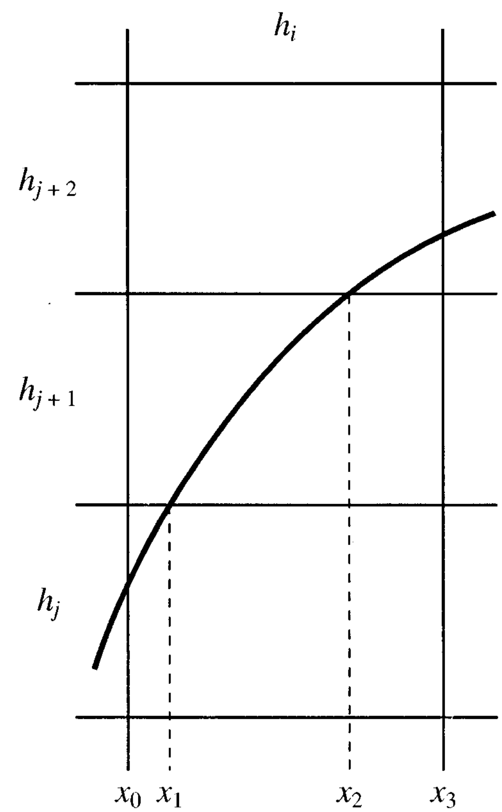

FIG. 1. Construction of a transfer matrix for a typical segment of a chaotic map. See text for the calculation of the corresponding matrix elements.

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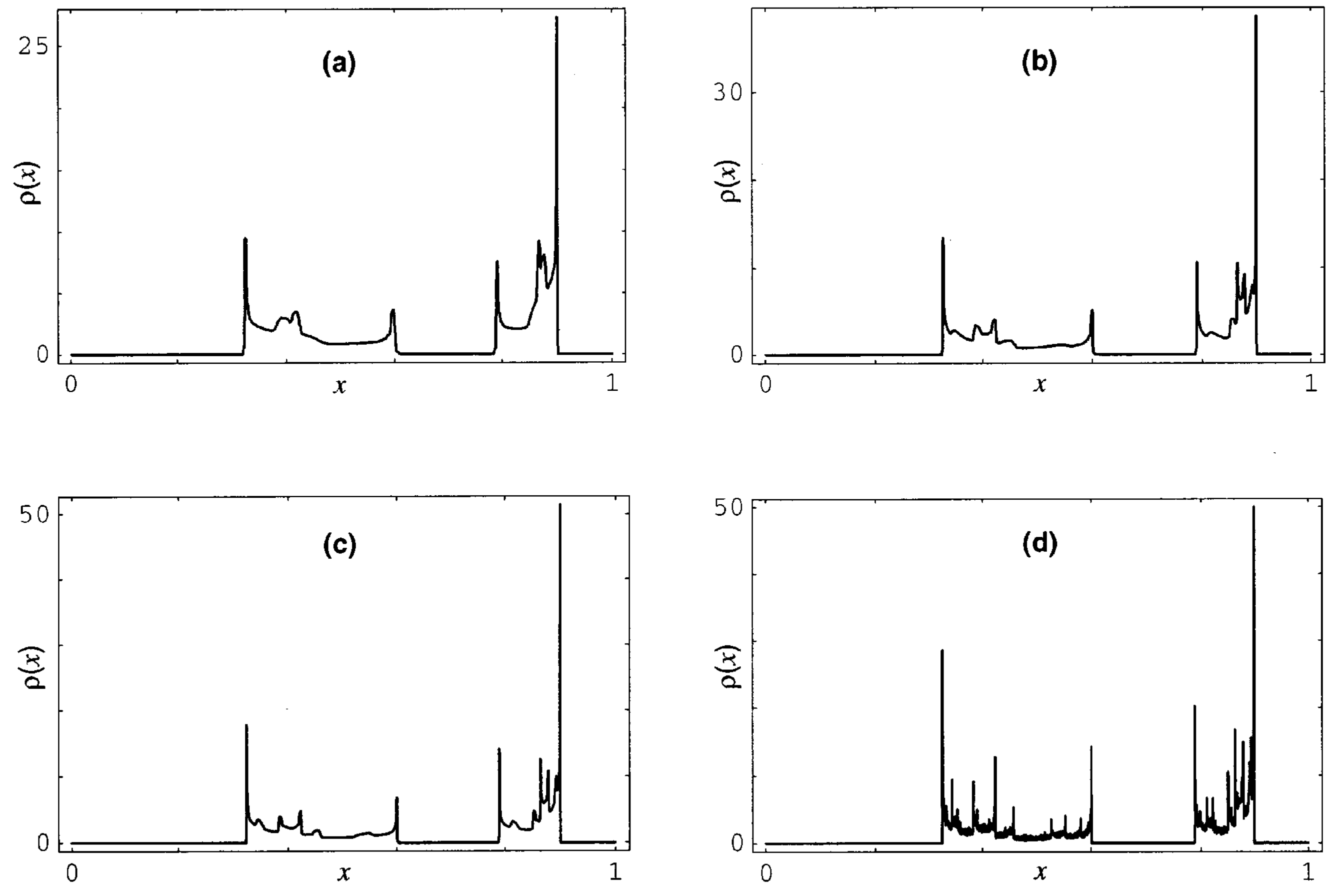

FIG. 2. Invariant distribution for the logistic map, $a=3.6$. (a)–(c) Transfer matrix (TM) method: 1000, 2000, and 4000 equal intervals respectively; (d) histogram method (see text for details).

logistic map in terms of a TM. In one version of the work, the TM has only one nonzero element per column; in the second version each interval is divided into a *finite* number of subintervals, which are used to express the transition probabilites as rational numbers, not as arbitrary real numbers as is done in the present work. The TM was used to follow the evolution of any initial distribution function. While the IM was not found directly, it was shown in [6] that it can be obtained as an asymptotic result.

The functional equation for the IM (see first paragraph) can now be written in vector form to find **r**, the vector representation of the IM. The equation now reads $\mathbf{Tr} = \mathbf{r}$, i.e., **r** is just the eigenvector of **T** corresponding to the eigenvalue $\Lambda=1$. This can be rewritten as $(\mathbf{T}-\mathbf{I})\,\mathbf{r} = \mathbf{0}$, where **I** is the identity matrix. Thus, finding the IM reduces to determining the kernel of $(\mathbf{T}-\mathbf{I})$. This may be achieved with the aid of any standard equation solver [7].

First we consider a simple analytical example: we calculate the IM for the Bernoulli shift, $x_{n+1}=2x_n$, mod (1). If we consider a uniform partition of the unit interval [0,1] into m intervals (m a large even number, with $n=m/2$), the nonzero elements of **T** are

$$T_{1,1}=T_{2,1}=T_{3,2}=T_{4,2}=\cdots=T_{m,n}=T_{1,n+1}=T_{2,n+1}=T_{3,n+2}=T_{4,n+2}=\cdots=T_{m,m}=\frac{1}{2}. \quad (1)$$

It can be easily verified that the vector $\mathbf{r} = (1/m, 1/m, \ldots)$ is the normalized eigenvector corresponding to $\Lambda=1$. In other words the IM is uniform, as is well known for this map. The reason one obtains the exact result is that the assumption [4] of locally uniform IM happens to be satisfied in this map.

Next we consider a more complicated example, the logistic map, already defined above. Many physical systems seem to be in the same universality class as the logistic map (see [8] for a review). For most values of a in the interval $3.57 \leqslant a \leqslant 4$ the IM has defied analytical calculation—the richness of structure of the functions can be seen in Fig. 2. A few facts about the logistic map's IM are known [9–11], for example that the peaks correspond to successive iterates of the critical point, $x=1/2$.

The logistic map's IM can be easily found for arbitrary values of a with the histogram method. There is ample numerical evidence for ergodicity (see [12] for an elementary discussion of this concept) in this map: the iterates of all initial conditions yield the same IM for each value of a. The histogram method, however, has two possible sources of error. One is that a number of iterations of the initial condition must be discarded to ensure that the trajectory is on the attractor, and that the sample over which the IM is estimated is meaningful. There is no *a priori* way to know how many iterations to throw away. The second is a statistical sampling error. For example with $a=3.8$ the IM shows an isolated peak at $x \sim 0.4319$, corresponding to the eighteenth iterate of

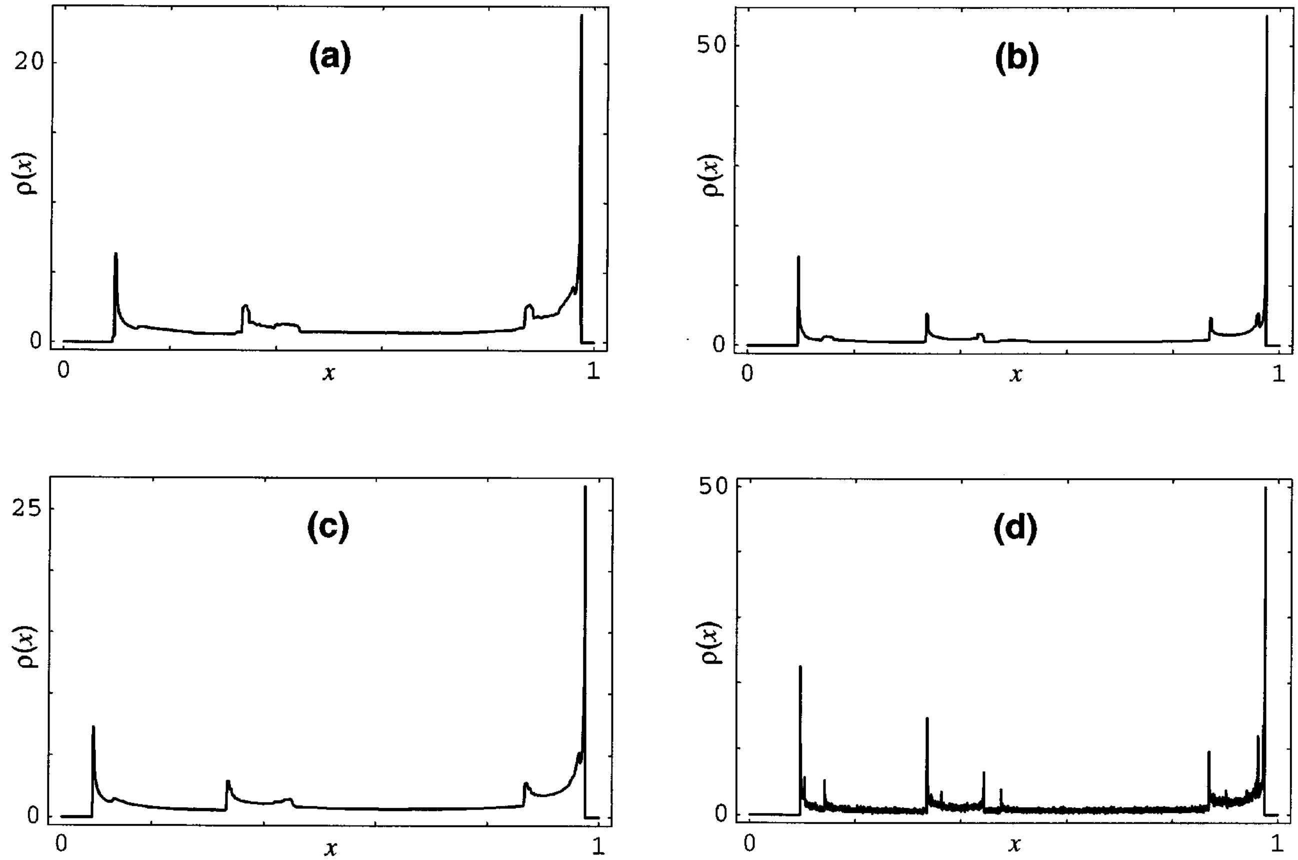

FIG. 3. Invariant distribution for the logistic map, $a=3.9$. TM method: (a) and (b) 1000 and 4000 equal intervals, respectively; (c) 1102 intervals, with higher resolution near the peaks, as explained in text; (d) histogram method: numbers same as in Fig. 2(d).

the critical point. We collected statistics over 20 independent trajectories of 10^4 points, not including a transient. The bin for the interval [0.40, 0.45] had a probability of 0.0281 ± 0.0016, or a relative error of about 5%; see also Fig. 2(d) for numerical evidence of noise when the histogram method is used. Despite these problems, the results of the histogram method seem to be reliable if long enough transients are discarded and large enough samples are used, and therefore we will use them for comparison.

Purely periodic cases in which the IM has a few sharp peaks have been considered in [6]. For $a=3.6$ the IM consists of two disjoint segments (band chaos): trajectories oscillate between the two bands at successive time steps; see also the bifurcation diagram in [9]. In Fig. 2 we compare TM and histogram results for this value of a. Parts (a)–(c) were obtained with the TM method with uniform intervals, $m=1000$, 2000, and 4000, respectively. In part (d) we used the histogram method: 10^4 points were discarded and 10^5 were used to collect statistics over 4000 bins. All results are normalized as probability densities. Three features can be seen in the TM graphs. One is that the holes in the IM emerge correctly in the column vector **r**. Another is that existing peaks grow while new ones form in the IM as m increases. Convergence to the true IM is expected to be slow: we are approximating a sharply peaked function with terraces of equal width. In the next paragraph we show how to partly circumvent this problem. Finally, in the flat regions between peaks the TM results are free of the statistical error observed in Fig. 2(d).

In Fig. 3 we show a case of full chaos with relatively few peaks. It illustrates the advantages of unequal intervals. Figures 3(a) and (b) show the results of 1000 and 4000 equal intervals respectively. In 3(c) we have constructed a matrix in which the intervals around the peaks observed in 3(a) were finely divided, in order to locate the peaks more precisely. In particular, the intervals [0.08,0.16], [0.32,0.36], [0.4,0.44], and [0.84,1.0] were divided into smaller intervals of 4×10^{-4}. The total number of intervals in this case is 1102. We see that the resolution increases considerably when compared with the case of 1000 intervals, and is comparable with the case of 4000 equal intervals. Figure 3(d), obtained with the histogram method with numbers as Fig. 2(d), is shown for comparison.

In this paper we have developed a direct method for calculating the invariant measure of chaotic maps. The formulation is based on constructing a transfer matrix and finding a particular eigenvector; it is general enough to apply to dissipative or Hamiltonian maps in one or several dimensions. The number of intervals m is limited not by time, but by memory [13].

The IMs we have found for the logistic map agree with those obtained with the histogram method; this provides independent evidence for the ergodicity of the logistic map. The main advantage of the TM method over the standard histogram method is the absence of sampling errors: for example, compare the smoothness of the flat parts of Figs. 2(c) and 2(d). The main disadvantage of the method is the assumption of a locally uniform IM within each interval, which smooths out the resulting IM. However, our results indicate that increasing the number of intervals results in systematic convergence to the true IM. This convergence can be further enhanced by increasing the resolution around the observed peaks, as is shown in Fig. 3(c).

The method of this paper can be useful in a realistic experimental situation: when the IM is known, but the full time series is not. In that case one could surmise certain properties of T (continuity, number of maxima or minima, maximum or average slope), and propose and refine a transfer matrix that is consistent with the experimental IM. We hope that this work will lead to better understanding and identification of real systems described by iterated maps.

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[4] Other than the discretization of space, this is the only approximation in our method. In reality the IM does vary within an interval. The observed result, when compared with the histogram method, is a smoothing out of the resulting IM.

[5] We have verified numerically that the approximation in the previous footnote holds in the calculations of [2].

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[7] We have found in all the cases we have tried that ($\mathbf{T}-\mathbf{I}$) is a singular matrix with a kernel given by a single nonzero vector, which corresponds to the invariant distribution. We have used MATHEMATICA for small values of m, and EISPACK for large values of m.

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